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On the Inversion of the Extended Hankel Transformation*

P. HEYWOOD

University of Edinburgh, Edinburgh, Scotland

AND

P. G. ROONEY

University of Toronto, Toronto, Ontario, Canada

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1. INTRODUCTION

The spaces $\mathscr{L}_{\mu,p}$ are defined in [8] for $1 \leqslant p \leqslant \infty$, $\mu \in \mathbb{R}$, to consist of those complex valued Lebesgue measurable functions f on $(0, \infty)$ such that $\|f\|_{\mu,p} < \infty$, where

$$\|f\|_{\mu,p} = \left\{ \int_0^\infty |x^\mu f(x)|^p \, dx/x \right\}^{1/p}, \qquad 1 \leqslant p < \infty,$$

$$= \operatorname*{ess\,sup}_{x>0} |x^\mu f(x)|, \qquad p = \infty.$$

We denote by $\mathbb{C}_0$ the collection of functions continuous and compactly supported on $(0, \infty)$, and by $[X, Y]$ the collection of bounded linear transformations from the Banach space X to the Banach space Y, $[X, X]$ being abbreviated to $[X]$.

In an earlier paper by one of us [6], the extended Hankel transformation was studied on the $\mathscr{L}_{\mu,p}$ spaces. The transformation is defined as follows. For $\nu \in \mathbb{R}$, k a non-negative integer, and $x > 0$, let

$$J_{\nu,k}(x) = \sum_{n=k}^{\infty} (-1)^n x^{\nu+2n} / (2^{\nu+2n} n! \, \Gamma(\nu+n+1)).$$

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Clearly $J_{\nu,0}(x)$ is the Bessel function $J_\nu(x)$; $J_{\nu,k}(x)$ is often referred to as a cut Bessel function. The extended Hankel transformation H_ν is defined for $f \in \mathbb{C}_0$ and for $\nu \in \mathbb{R}$, $\nu \neq -1, -3, \ldots$, by

$$(H_\nu f)(x) = \int_0^\infty (xt)^{1/2} J_{\nu,l}(xt)\, f(t)\, dt, \tag{1.1}$$

where $l = l_\nu$ is the least non-negative integer such that $\nu + 2l > -1$. In [6] it was shown that if $1 < p < \infty$, $\gamma(p) \leqslant \mu < \nu + 2l + 3/2$, where

$$\gamma(p) = \max(1/p, 1/p'), \qquad p' = p/(p-1)$$

and $l = l_\nu$, then $H_\nu \in [\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,q}]$ for all $q \geqslant p$ such that $q'^{-1} \leqslant \mu$.

However, in [6] no representation of the transformation was given nor were any inversion formulae proved for it; also, the boundedness of the transformation on $\mathscr{L}_{\mu,1}$ was not studied, and it is the object of the present paper to repair these omissions. The boundedness on $\mathscr{L}_{\mu,1}$ is studied in Section 2, while Section 3 is devoted to some lemmas preliminary to the later sections. In Section 4 we will find a representation for the transformation, while in Section 5 we will find formulas for the inversion of the transformation on $\mathscr{L}_{\mu,p}$ except when $\mu = -(\nu + 2l) + 3/2$. It will be noted that in [6] we were able to characterize the range of the transformation on $\mathscr{L}_{\mu,p}$ except when $\mu = -(\nu + 2l) + 3/2$. It transpires that this is a special case, and we shall study it in Section 6, characterizing the range and finding inversion formulae in this case.

Throughout this paper we shall assume that ν is a fixed real number, less than minus one and not equal to $-3, -5, \ldots$, and l will always mean l_ν as defined above. Almost all of our results are also valid for $\nu > -1$, though the proofs may be a bit different then, but for $\nu > -1$ representation theorems and inversions theorems and the boundedness on $\mathscr{L}_{\mu,p}$ were proved elsewhere, see [7, 3], so we shall not prove them here.

One of our main tools will be the Mellin transformation $\mathscr{M}$, whose definition and theory are summarized in [8, Sect. 2] and we shall use those results frequently. Also in [5] it is shown that if $f \in \mathscr{L}_{1/2,2} = L_2(0, \infty)$, then for $\operatorname{Re} s = 1/2$

$$(\mathscr{M} H_\nu f)(s) = m_\nu(s)(\mathscr{M} f)(1-s), \tag{1.2}$$

where

$$m_\nu(s) = 2^{s-1/2}\Gamma((\nu + s + 1/2)/2)/\Gamma((\nu - s + 3/2)/2). \tag{1.3}$$

In fact, it is easy to see that (1.2) is valid, with $\operatorname{Re} s = 1 - \mu$, if $f \in \mathscr{L}_{\mu,p}$ where $1 < p \leqslant 2$, $\gamma(p) \leqslant \mu < \nu + 2l + 3/2$, for then both sides of (1.2) represent bounded linear transformations of $\mathscr{L}_{\mu,p}$ into $L_{p'}(-\infty, \infty)$; we will use (1.2) frequently.

2. BOUNDEDNESS

The boundedness of H_ν on $\mathscr{L}_{\mu,1}$ is given by the following theorem.

THEOREM 2.1. *If* $1 \leqslant \mu \leqslant \nu + 2l + 3/2$, *then* $H_\nu \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,\infty}]$, *and if* $f \in \mathscr{L}_{\mu,1}$, $H_\nu f$ *is given by* (1.1). *If* $1 < \mu < \nu + 2l + 3/2$, *then for all* p, $1 \leqslant p \leqslant \infty$, $H_\nu \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,p}]$.

Proof. Clearly $J_{\nu,l}(x) \sim L_\nu x^{\nu+2l}$ as $x \to 0+$, where L_ν is a constant. Also, from [1, 7.13.1(13)], since $\nu + 2l - 2 < -1$, $J_{\nu,l}(x) \sim (2/(\pi x))^{1/2} \cos(x - (\nu + 1/2)\pi)$ as $x \to \infty$. Thus since, if $1 \leqslant \mu \leqslant \nu + 2l + 3/2$, then $\nu + 2l \geqslant -1/2$, there is a constant K_ν so that

$$|J_{\nu,l}(x)| \leqslant K_\nu \cdot \min(x^{\nu+2l}, x^{-1/2}) \leqslant K_\nu x^{\mu-3/2}.$$

But then if $f \in \mathbb{C}_0$, $x > 0$, and $1 \leqslant \mu \leqslant \nu + 2l + 3/2$,

$$\begin{aligned} |(H_\nu f)(x)| &\leqslant \int_0^\infty (xt)^{1/2}\, |J_{\nu,l}(xt)|\, |f(t)|\, dt \\ &\leqslant K_\nu x^{\mu-1} \int_0^\infty t^{\mu-1} |f(t)|\, dt = K_\nu x^{\mu-1} \|f\|_{\mu,1}. \end{aligned}$$

Thus

$$\|H_\nu f\|_{1-\mu,\infty} = \operatorname*{ess\,sup}_{x>0} x^{1-\mu} |(H_\nu f)(x)| \leqslant K_\nu \|f\|_{\mu,1}$$

for $f \in \mathbb{C}_0$, so that H_ν can be extended to $\mathscr{L}_{\mu,1}$ as a member of $[\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,\infty}]$ for $1 \leqslant \mu \leqslant \nu + 3/2$, and is clearly given by (1.1) on $\mathscr{L}_{\mu,1}$.

Let $M = \int_0^\infty x^{1/2-\mu} |J_{\nu,l}(x)|\, dx$. Then, from the asymptotic behavior of $J_{\nu,l}(x)$ near zero and infinity noted above, $M < \infty$ if and only if $\int_0^\delta x^{\nu+2l-\mu+1/2}\, dx$ and $\int_R^\infty x^{-\mu}\, dx$ are finite, where $0 < \delta < R < \infty$, that is, if and only if $1 < \mu < \nu + 2l + 3/2$. But then if $f \in \mathscr{L}_{\mu,1}$,

$$\begin{aligned} \|H_\nu f\|_{1-\mu,1} &= \int_0^\infty x^{1-\mu} |(H_\nu f)(x)|\, dx/x \\ &\leqslant \int_0^\infty x^{-\mu}\, dx \int_0^\infty (xt)^{1/2}\, |J_{\nu,l}(xt)|\, |f(t)|\, dt \\ &= \int_0^\infty t^{1/2} |f(t)|\, dt \int_0^\infty x^{1/2-\mu} |J_{\nu,l}(tx)|\, dx = M \|f\|_{\mu,1}, \end{aligned}$$

so that if $1 < \mu < \nu + 2l + 3/2$, $H_\nu \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,1}]$. Finally, by interpolation, using [9, Theorem 2], if $1 < \mu < \nu + 2l + 3/2$, $1 \leqslant p \leqslant \infty$, $H_\nu \in [\mathscr{L}_{\mu,1}, \mathscr{L}_{1-\mu,p}]$.

3. Some Preliminary Lemmas

Lemma 3.1. *For $x>0$ let*

$$I_1=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s}m_\nu(s)/(3/2-\nu-s)\,ds,$$

$$I_2=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s}m_\nu(s)/((7/2-\nu-s)(3/2-\nu-s))\,ds,$$

and

$$I_3=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s}m_\nu(s)/((11/2-\nu-s)\times(7/2-\nu-s)(3/2-\nu-s))\,ds,$$

the integrals being taken in the principal value sense at infinity. Then:

(i) *If $\sigma<3/2$,*

$$I_1=-x^{-1/2}J_{\nu-1,l+1}(x)$$

for $-(\nu+2l)-1/2<\sigma<-(\nu+2l)+3/2$, *and*

$$I_1=x^{-1/2}J_{\nu-1,l}(x)$$

for $-(\nu+2l)+3/2<\sigma<-(\nu+2l)+7/2$.

(ii) *If $\sigma<5/2$ then*

$$I_2=x^{-3/2}J_{\nu-2,l+2}(x)$$

for $-(\nu+2l)-1/2<\sigma<-(\nu+2l)+3/2$, *and*

$$I_2=x^{-3/2}J_{\nu-2,l+1}(x)$$

for $-(\nu+2l)+3/2<\sigma<-(\nu+2l)+7/2$.

(iii) *If $\sigma<7/2$,*

$$I_3=-x^{-5/2}J_{\nu-3,l+3}(x)$$

for $-(\nu+2l)-1/2<\sigma<-(\nu+2l)+3/2$, *and*

$$I_3=-x^{-5/2}J_{\nu-3,l+2}(x)$$

for $-(\nu+2l)+3/2<\sigma<-(\nu+2l)+7/2$.

Proof. From [5, (2.4)], for $x>0$, $\xi<1$ and $-(\lambda+2k)<\xi<-(\lambda+2k)+2$, where k is a non-negative integer,

$$\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} x^{-s}2^{s-1}\Gamma((\lambda+s)/2)/\Gamma((\lambda-s)/2+1)\,ds = J_{\lambda,k}(x),$$

where the integral is taken in the principal value sense at infinity. Hence setting $\lambda=\nu-1$ and $k=l+1$, where of course $l=l_\nu$, if $-(\nu+2l)-1<\xi<-(\nu+2l)+1$ and $\xi<1$,

$$\begin{aligned}x^{-1/2}J_{\nu-1,l+1}(x) &= \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} x^{-(s+1/2)}2^{s-1}\\ &\quad\times\Gamma((\nu+s-1)/2)/\Gamma((\nu-s-1)/2+1)\,ds\\ &= \frac{1}{2\pi i}\int_{\xi+1/2-i\infty}^{\xi+1/2+i\infty} x^{-s}2^{s-3/2}\\ &\quad\times\Gamma((\nu+s-3/2)/2)/\Gamma((\nu-s-1/2)/2+1)\,ds.\end{aligned}$$

However, using $\Gamma(z+1)=z\Gamma(z)$,

$$\begin{aligned}&2^{s-3/2}\Gamma((\nu+s-3/2)/2)/\Gamma((\nu-s-1/2)/2+1)\\ &\quad=2^{s-1/2}\Gamma((\nu+s+1/2)/2)/((\nu+s-3/2)\Gamma((\nu-s+3/2)/2))\\ &\quad=-m_\nu(s)/(3/2-\nu-s).\end{aligned}$$

Thus, writing σ for $\xi+1/2$, if $-(\nu+2l)-1/2<\sigma<-(\nu+2l)+3/2$ and $\sigma<3/2$, $I_1=-x^{-1/2}J_{\nu-1,l+1}(x)$.

The proof in the other cases is similar.

LEMMA 3.2. *For $x>0$ let*

$$\begin{aligned}g_x(t)&=t^{1/2-\nu}, &&0<t<x,\\ &=0, &&t>x.\end{aligned}$$

Then $g_x\in\mathscr{L}_{\mu,p}$ if $1\leqslant p<\infty$ and $\mu>\nu-1/2$, or if $p=\infty$ and $\mu\geqslant\nu-1/2$. Also

$$(H_\nu g_x)(t)=-x^{1-\nu}t^{-1/2}J_{\nu-1,l+1}(xt).$$

Proof. If $1\leqslant p<\infty$,

$$\|g_x\|_{\mu,p}=\left\{\int_0^x (t^{\mu+1/2-\nu})^p\,dt/t\right\}^{1/p}<\infty$$

if $\mu > \nu - 1/2$, while if $\mu \geqslant \nu - 1/2$,

$$\|g_x\|_{\mu,\infty} = \operatorname*{ess\ sup}_{0<t<x} t^{\mu+1/2-\nu} = x^{\mu+1/2-\nu} < \infty.$$

Since $\mu > \nu + 1/2$, $g_x \in \mathscr{L}_{\mu,1}$, and thus,

$$(H_\nu g_x)(t) = \int_0^x u^{1/2-\nu}(tu)^{1/2} J_{\nu,l}(tu)\, du$$

$$= t^{1/2} \int_0^x u^{1-\nu} J_{\nu,l}(tu)\, du.$$

But, from [5, (2.1)], $u^{1-\nu}J_{\nu,l}(tu) = -t^{-1}(d/du)\, u^{1-\nu}J_{\nu-1,l+1}(tu)$, and thus

$$(H_\nu g_x)(t) = -x^{1-\nu}t^{-1/2}J_{\nu-1,l+1}(xt).$$

LEMMA 3.3. *If* $f \in \mathscr{L}_{\mu,p}$ *and* $g \in \mathscr{L}_{\mu,p'}$, *where* $1 < p < \infty$, $\gamma(p) \leqslant \mu < \nu + 2l + 3/2$, *then*

$$\int_0^\infty f(x)(H_\nu g)(x)\, dx = \int_0^\infty (H_\nu f)(x)\, g(x)\, dx. \tag{3.1}$$

Proof. If f and g are in $\mathscr{L}_{1/2,2}$, the result follows from [5, (4.1)]. But, if $f \in \mathscr{L}_{\mu,p}$ and $g \in \mathscr{L}_{\mu,p'}$,

$$\left|\int_0^\infty f(x)(H_\nu g)(x)\, dx\right| \leqslant \|f\|_{\mu,p} \cdot \|H_\nu g\|_{1-\mu,p'}$$

$$\leqslant K\|f\|_{\mu,p} \cdot \|g\|_{\mu,p'},$$

where K is the norm of H_ν as a member of $[\mathscr{L}_{\mu,p'}, \mathscr{L}_{1-\mu,p'}]$, so that the left hand side of (3.1) is a bounded bilinear functional on $\mathscr{L}_{\mu,p} \times \mathscr{L}_{\mu,p'}$, as is the right hand side similarly, and thus (3.1) holds under the stated conditions.

4. REPRESENTATION

The following theorem gives a representation of H_ν. It will be noted that it is expressed in terms of $J_{\nu-1,l+1}$ whereas the representation for H_ν for $\nu > -1$, given in [7, Theorem 2.2], is in terms of $J_{\nu+1}$. We could obtain a representation theorem in terms of $J_{\nu+1,l}$, but we choose the form below since it transpires that the inversion formulae come out most simply in terms of $J_{\nu-k,l+k}$, where $k = 1, 2$, or 3.

THEOREM 4.1. *Suppose that $f \in \mathscr{L}_{\mu,p}$ where $1 \leqslant p < \infty$, $\gamma(p) \leqslant \mu < \nu + 2l + 3/2$. Then for almost all $x > 0$,*

$$(H_\nu f)(x) = -x^{\nu-1/2} \frac{d}{dx} x^{1/2-\nu} \int_0^\infty (xt)^{1/2} J_{\nu-1,l+1}(xt) f(t)\, dt/t.$$

Proof. Suppose first $f \in \mathbb{C}_0$. Then $f \in \mathscr{L}_{1/2,2}$ and from Lemma 3.2, $g_x \in \mathscr{L}_{1/2,2}$. Thus, from Lemma 3.3 and Lemma 3.2, if $x > 0$,

$$\begin{aligned}\int_0^x t^{1/2-\nu}(H_\nu f)(t)\, dt &= \int_0^\infty g_x(t)(H_\nu f)(t)\, dt\\ &= \int_0^\infty (H_\nu g_x)(t) f(t)\, dt\\ &= -x^{1/2-\nu} \int_0^\infty (xt)^{1/2} J_{\nu-1,l+1}(xt) f(t)\, dt/t. \qquad (4.1)\end{aligned}$$

But for each $x > 0$ both sides of (4.1) represent bounded linear functionals on $\mathscr{L}_{\mu,p}$ if $1 \leqslant p < \infty$, $\gamma(p) \leqslant \mu < \nu + 2l + 3/2$. For, if $f \in \mathscr{L}_{\mu,p}$, where $p > 1$, and $x > 0$, then, by Holder's inequality,

$$\begin{aligned}\left|\int_0^x t^{1/2-\nu}(H_\nu f)(t)\, dt\right| &= \left|\int_0^\infty g_x(t)(H_\nu f)(t)\, dt\right|\\ &\leqslant \|g_x\|_{\mu,p'} \cdot \|H_\nu f\|_{1-\mu,p}\\ &\leqslant \|g_x\|_{\mu,p'} \cdot K\|f\|_{\mu,p},\end{aligned}$$

where K is the norm of H_ν as a member of $[\mathscr{L}_{\mu,p}, \mathscr{L}_{1-\mu,p}]$, and

$$\begin{aligned}&\left|x^{1/2-\nu} \int_0^\infty (xt)^{1/2} J_{\nu-1,l+1}(xt) f(t)\, dt/t\right|\\ &\quad = \left|\int_0^\infty (H_\nu g_x)(t) f(t)\, dt\right|\\ &\quad \leqslant \|H_\nu g_x\|_{1-\mu,p'} \cdot \|f\|_{\mu,p} \leqslant K'\|g_x\|_{\mu,p'} \cdot \|f\|_{\mu,p},\end{aligned}$$

where K' is the norm of H_ν as a member of $[\mathscr{L}_{\mu,p'}, \mathscr{L}_{1-\mu,p'}]$, since $\mu \geqslant \gamma(p) \geqslant 1/2 > \nu - 1/2$ and, from Lemma 3.2, $g_x \in \mathscr{L}_{\mu,p'}$. If $f \in \mathscr{L}_{\mu,1}$, then from Theorem 2.1, $H_\nu f \in \mathscr{L}_{1-\mu,\infty}$ and thus

$$\begin{aligned}\left|\int_0^x t^{1/2-\nu}(H_\nu f)(t)\, dt\right| &= \left|\int_0^\infty g_x(t)(H_\nu f)(t)\, dt\right|\\ &\leqslant \|g_x\|_{\mu,1} \cdot \|H_\nu f\|_{1-\mu,\infty}\end{aligned}$$

$\leqslant K\|g_x\|_{\mu,1} \cdot \|f\|_{\mu,1}$, since from Lemma 3.2, $g_x \in \mathscr{L}_{\mu,1}$. Also if $\phi_x(t) = (xt)^{1/2} J_{\nu-1,l+1}(xt)/t$, it is easy to see that $\phi_x \in \mathscr{L}_{1-\mu,\infty}$, and thus

$$\left| x^{1/2-\nu} \int_0^\infty (xt)^{1/2} J_{\nu-1,l+1}(xt) f(t)\, dt/t \right|$$
$$\leqslant x^{1/2-\nu} \|\phi_x\|_{1-\mu,\infty} \cdot \|f\|_{\mu,1}.$$

Hence (4.1) holds on $\mathscr{L}_{\mu,p}$ for the values of μ and p prescribed, and differentiating both sides with respect to x the result follows.

5. Inversion

The inversion theory of H_ν on $\mathscr{L}_{\mu,p}$ is different according as μ is less than, larger than, or equal to $-(\nu+2l)+3/2$. The first two of these cases will be treated in this section, the first case in Theorem 5.1 and the second in Theorem 5.2 below. The third case, which is much harder, will be treated in the next section.

Theorem 5.1. *Suppose $f \in \mathscr{L}_{\mu,p}$, where $1 \leqslant p < \infty$, $\gamma(p) \leqslant \mu < \nu + 2l + 3/2$, and $\mu < -(\nu+2l)+3/2$. Then for almost all $x > 0$,*

$$f(x) = -x^{\nu+1/2} \left\{ x^{-1} \frac{d}{dx} \right\}^3 x^{5/2-\nu} \times \int_0^\infty (xt)^{1/2} J_{\nu-3,l+3}(xt)(H_\nu f)(t)\, dt/t^3. \tag{5.1}$$

If, in addition, $\mu < 2$, then for almost all $x > 0$,

$$f(x) = x^{\nu+1/2} \left\{ x^{-1} \frac{d}{dx} \right\}^2 x^{3/2-\nu} \times \int_0^\infty (xt)^{1/2} J_{\nu-2,l+2}(xt)(H_\nu f)(t)\, dt/t^2, \tag{5.2}$$

while if, in addition, $\mu < 1$, then for almost all $x > 0$,

$$f(x) = -x^{\nu-1/2} \frac{d}{dx} x^{1/2-\nu} \int_0^\infty (xt)^{1/2} J_{\nu-1,l+1}(xt)(H_\nu f)(t)\, dt/t. \tag{5.3}$$

Proof. For $x > 0$ let

$$h_x(t) = t^{1/2-\nu}(x^2-t^2)^2/8, \qquad 0 < t < x,$$
$$= 0, \qquad t > x.$$

Then as for g_x earlier, $h_x \in \mathscr{L}_{\mu,p}$ if $1 \leqslant p < \infty$, $\mu > \nu - 1/2$, or if $p = \infty$, $\mu \geqslant \nu - 1/2$. In particular $h_x \in \mathscr{L}_{\mu,2}$. Also $(\mathscr{M}h_x)(s) = x^{s-\nu+9/2}/((s-\nu+9/2)(s-\nu+5/2)(s-\nu+1/2))$, $\operatorname{Re} s = \mu$, by an elementary calculation.

Let $\phi_x(t) = -x^{3-\nu}t^{-5/2}J_{\nu-3,l+3}(xt)$. Then from Lemma 3.1, since $-(\nu+2l)-1/2 < 1/2 \leqslant \mu < -(\nu+2l)+3/2$,

$$\phi_x(t) = x^{11/2-\nu}\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} t^{-s}x^{-s} \times m_\nu(s)/((11/2-\nu-s)(7/2-\nu-s)(3/2-\nu-s))\, ds. \tag{5.4}$$

But from [1, 1.18(6)], $|m_\nu(\mu+i\tau)| \sim |\tau|^{\mu-1/2}$ as $|\tau| \to \infty$, and consequently since $\mu < \nu+2l+3/2 < 5/2$, the integrand in (5.4) is in $L_2(-\infty,\infty)$ on the line $\operatorname{Re} s = \mu$. However, the Mellin transformation is a one-to-one mapping of $\mathscr{L}_{\mu,2}$ onto $L_2(-\infty,\infty)$ on the line $\operatorname{Re} s = \mu$, and thus $\phi_x \in \mathscr{L}_{\mu,2}$ and

$$(\mathscr{M}\phi_x)(s) = x^{11/2-\nu-s}m_\nu(s)/((11/2-\nu-s)(7/2-\nu-s)(3/2-\nu-s)).$$

Hence if $\operatorname{Re} s = 1-\mu$,

$$\begin{aligned}(\mathscr{M}H_\nu\phi_x)(s) &= m_\nu(s)\cdot(\mathscr{M}\phi_x)(1-s)\\ &= x^{s-\nu+9/2}m_\nu(s)\,m_\nu(1-s)/((s-\nu+9/2)\\ &\qquad \times(s-\nu+5/2)(s-\nu+1/2)) = (\mathscr{M}h_x)(s),\end{aligned}$$

so that

$$H_\nu\phi_x = h_x.$$

But then if $f \in \mathscr{L}_{\mu,2}$, using Lemma 3.3,

$$\begin{aligned}&-x^{5/2-\nu}\int_0^\infty (xt)^{1/2}J_{\nu-3,l+3}(xt)(H_\nu f)(t)\,dt/t^3\\ &=\int_0^\infty \phi_x(t)(H_\nu f)(t)\,dt\\ &=\int_0^\infty (H_\nu\phi_x)(t)\,f(t)\,dt\\ &=\int_0^\infty h_x(t)\,f(t)\,dt\\ &=\int_0^x t^{1/2-\nu}(x^2-t^2)^2 f(t)\,dt/8.\end{aligned} \tag{5.5}$$

But the expressions at either end of (5.5) represent bounded linear functionals on $\mathscr{L}_{\mu,p}$, the expression at the end since, from above, $h_x \in \mathscr{L}_{1-\mu,p'}$

if $1-\mu>\nu-1/2$, which is obviously true since $\mu<5/2$. For the beginning expression, notice that $\phi_x(t)=O(t^{\nu+2l+1/2})$ as $t\to 0+$, and $\phi_x(t)=O(\max(t^{-3}, t^{\nu+2l-3/2}))$ as $t\to\infty$. Thus since $-(\nu+2l)-1/2<1/2\leqslant\mu<-(\nu+2l)+3/2$, and $\mu<\nu+2l+3/2$ implies $\mu<5/2<3$, $\phi_x\in\mathscr{L}_{\mu,p'}$. Hence, since $H_\nu f\in\mathscr{L}_{1-\mu,p}$ the beginning expression in (5.5) also represents a bounded linear functional on $\mathscr{L}_{\mu,p}$. Thus (5.5) holds for $f\in\mathscr{L}_{\mu,p}$. But a trivial calculation shows that

$$\left\{x^{-1}\frac{d}{dx}\right\}^3\int_0^x t^{1/2-\nu}(x^2-t^2)^2 f(t)\,dt/8=x^{-1/2-\nu}f(x) \qquad \text{a.e.,}$$

and so differentiating (5.5), (5.1) follows.

The proof of (5.2) is similar, replacing h_x by k_x where

$$\begin{aligned} k_x(t)&=x^{1/2-\nu}(x^2-t^2)/2, && 0<t<x,\\ &=0, && t>x, \end{aligned}$$

using Lemma 3.1 to show

$$\begin{aligned} &x^{2-\nu}t^{-3/2}J_{\nu-2,l+2}(xt)\\ &\quad=x^{7/2-\nu}\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}t^{-s}x^{-s}m_\nu(s)/((7/2-\nu-s)(3/2-\nu-s))\,ds, \end{aligned}$$

and noticing that if $\mu<2$, the integrand in this integral is in $L_2(-\infty,\infty)$ on $\operatorname{Re} s=\mu$.

The proof of (5.3) is also similar.

THEOREM 5.2. *Suppose $f\in\mathscr{L}_{\mu,p}$, where $1\leqslant p<\infty$, $\gamma(p)\leqslant\mu<\nu+2l+3/2$, and $\mu>-(\nu+2l)+3/2$. Then for almost all $x>0$,*

$$\begin{aligned} f(x)=&-x^{\nu+1/2}\left\{x^{-1}\frac{d}{dx}\right\}^3 x^{5/2-\nu}\\ &\times\int_0^\infty (xt)^{1/2}J_{\nu-3,l+2}(xt)(H_\nu f)(t)\,dt/t^3. \end{aligned} \tag{5.6}$$

If, in addition, $\mu<2$, then for almost all $x>0$,

$$\begin{aligned} f(x)=&\,x^{\nu+1/2}\left\{x^{-1}\frac{d}{dx}\right\}^2 x^{3/2-\nu}\\ &\times\int_0^\infty (xt)^{1/2}J_{\nu-2,l+1}(xt)(H_\nu f)(t)\,dt/t^2, \end{aligned} \tag{5.7}$$

while if, in addition, $\mu < 1$, *then for almost all* $x > 0$

$$f(x) = -x^{\nu-1/2}\frac{d}{dx}x^{1/2-\nu}\int_0^\infty (xt)^{1/2} J_{\nu-1,l}(xt)(H_\nu f)(t)\,dt/t. \qquad (5.8)$$

Proof. The proof is practically the same as that of Theorem 5.1, but replacing ϕ_x, for example, by ψ_x where

$$\psi_x(t) = -x^{3-\nu}t^{-5/2}J_{\nu-3,l+2}(xt),$$

and using the value of I_3 given in Lemma 3.1 for the μ range in this theorem.

6. Inversion—Special Case

In this section we shall study the inversion of H_ν in the special case when $\mu = 3/2 - \nu - 2l$. In [6] we showed that, except in this special case, the range of H_ν on $\mathscr{L}_{\mu,p}$ was the same as the range of H_η where $\eta = |\nu + 2l|$, at least for $1 < p < \infty$. Here we shall show first that the range of H_ν on $\mathscr{L}_{\mu,p}$ for the special value of μ is a proper subset of the range of H_η and we will characterize the subset. The reader should note also that when $\mu = -(\nu + 2l) + 3/2$ and μ satisfies the conditions for the boundedness of H_ν, that is, $\gamma(p) \leqslant \mu < \nu + 2l + 3/2$, then $\nu + 2l > 0$, and $\mu < 3/2$. We show that then the formulas given for the inversion when $\mu < 1$ and when $\mu < 2$ in Theorems 5.1 and 5.2 work for the special value of μ provided one takes the correct limit of integration to be an arrow limit. We first need some lemmas.

Lemma 6.1. *Suppose* $1 \leqslant p \leqslant \infty$ *and let* r *be an integer such that* $1 \leqslant r \leqslant l - 1$. *Then*

(i) *if* $\mu > \nu + 2l - 5/2$, *there is a transformation* $T_r \in [\mathscr{L}_{\mu,p}]$ *such that if* $f \in \mathscr{L}_{\mu,p}$, *where* $1 \leqslant p \leqslant 2$,

$$(\mathscr{M}T_r f)(s) = ((\nu + s + 2r - 3/2)/(\nu - s + 2r - 1/2))(\mathscr{M}f)(s), \qquad \operatorname{Re} s = \mu;$$

(ii) *if* $\mu < 3/2 - \nu - 2l$, *there is a transformation* $T \in [\mathscr{L}_{\mu,p}]$ *such that if* $f \in \mathscr{L}_{\mu,p}$, *where* $1 \leqslant p \leqslant 2$,

$$(\mathscr{M}Tf)(s) = ((\nu - s + 2l - 1/2)/(\nu + s + 2l - 3/2))(\mathscr{M}f)(s), \qquad \operatorname{Re} s = \mu.$$

Proof. The results follows from [4, Theorem 3.1] taking, in the notation of that theorem, $T_r = -Q_{\alpha,\beta}$ with $\alpha = 3/2 - \nu - 2r$, $\beta = \nu + 2r - 1/2$, and $T = -P_{\lambda,\zeta}$ with $\lambda = \nu + 2l - 1/2$, $\zeta = 3/2 - \nu - 2l$. The conditions for T_r and T to be in $[\mathscr{L}_{\mu,p}]$ are respectively $\mu > \nu + 2r - 1/2$, which is satisfied for all values of $r \leqslant l - 1$ if $\mu > \nu + 2l - 5/2$, and $\mu < 3/2 - \nu - 2l$.

LEMMA 6.2. *If* $1 \leqslant p < \infty$, *and* $\gamma(p) \leqslant \mu < \nu + 2l + 3/2$, *then on* $\mathscr{L}_{\mu, p}$,

$$H_\nu = TH_{\nu+2l}T_{l-1}T_{l-2} \cdots T_1. \tag{6.1}$$

Proof. For $r = 1, 2, ..., l-1$, $T_r \in [\mathscr{L}_{\mu, p}]$ since $\nu + 2l - 5/2 < -3/2$, while $\mu \geqslant \gamma(p) \geqslant 1/2$. From [8, Theorem 5.1], under our hypotheses, $H_{\nu+2l} \in [\mathscr{L}_{\mu, p}, \mathscr{L}_{1-\mu, p}]$, and $T \in [\mathscr{L}_{1-\mu, p}]$ by Lemma 6.1 since $\nu + 2l - 1/2 < 1/2 \leqslant \mu$ so that $1 - \mu < 3/2 - \nu - 2l$. Hence the right hand side of (6.1) is in $[\mathscr{L}_{\mu, p}, \mathscr{L}_{1-\mu, p}]$ as is the left hand side by [6, Theorem 1]. But if $f \in \mathscr{L}_{\mu, 2}$ then for $\operatorname{Re} s = \mu$, by (1.2) and Lemma 6.1,

$$\begin{aligned}
&(\mathscr{M}TH_{\nu+2l}T_{l-1}T_{l-2} \cdots T_1 f)(s)\\
&\quad = ((\nu - s + 2l - 1/2)/(\nu + s + 2l - 3/2))\\
&\qquad \times (\mathscr{M}H_{\nu+2l}T_{l-1}T_{l-2} \cdots T_1 f)(s)\\
&\quad = ((\nu - s + 2l - 1/2)/(\nu + s + 2l - 3/2))\\
&\qquad \times m_{\nu+2l}(s)(\mathscr{M}T_{l-1}T_{l-2} \cdots T_1 f)(1-s)\\
&\quad = m_{\nu+2l}(s)((\nu - s + 2l - 1/2)(\nu - s + 2l - 5/2)\\
&\qquad \cdots (\nu - s + 3/2)/((\nu + s + 2l - 3/2)\\
&\qquad \cdot (\nu + s + 2l - 7/2) \cdots (\nu + s + 1/2))) \cdot (\mathscr{M}f)(1-s)\\
&\quad = 2^{s-1/2}[\Gamma((\nu + 2l + s + 1/2)/2)/((\nu + s + 2l - 3/2)\\
&\qquad \cdot (\nu + s + 2l - 7/2) \cdots (\nu + s + 1/2))]/[\Gamma((\nu + 2l\\
&\qquad - s + 3/2)/2)/((\nu - s + 2l - 1/2)(\nu - s + 2l - 5/2)\\
&\qquad \cdots (\nu - s + 3/2))](\mathscr{M}f)(1-s)\\
&\quad = 2^{s-1/2}(\Gamma((\nu + s + 1/2)/2)/\Gamma((\nu - s + 3/2)/2))(\mathscr{M}f)(1-s)\\
&\quad = m_\nu(s)(\mathscr{M}f)(1-s)\\
&\quad = (\mathscr{M}H_\nu f)(s).
\end{aligned}$$

Hence, the two sides of (6.1) agree on $\mathscr{L}_{\mu,2}$, and since both sides of (6.1) are in $[\mathscr{L}_{\mu, p}]$, and $\mathbb{C}_0$ is dense in $\mathscr{L}_{\mu, p}$, (6.1) holds on $\mathscr{L}_{\mu, p}$.

THEOREM 6.3. *Suppose that* $f \in \mathscr{L}_{\mu, p}$ *where* $1 \leqslant p < \infty$, $\gamma(p) \leqslant \mu < \nu + 2l + 3/2$, *and* $\mu = 3/2 - \nu - 2l$. *Then the integral*

$$\int_{\to 0}^{\to \infty} t^{\nu + 2l - 3/2}(H_\nu f)(t)\, dt \tag{6.2}$$

converges and has the value zero.

Proof. If we denote $H_{\nu+2l}T_{l-1}T_{l-2}\cdots T_1 f$ by h, then from Lemma 6.2, $h \in \mathscr{L}_{1-\mu,p}$ and $(H_\nu f)(t) = (Th)(t)$ for almost all positive t. But as noted in the proof of Lemma 6.1, $T = -P_{\lambda,\zeta}$ where $\lambda = \nu + 2l - 1/2$ and $\zeta = 3/2 - \nu - 2l$. Integral (6.2) now follows from [4, Theorem 3.3] since $\lambda = 1 - \mu$.

COROLLARY 6.4. *Suppose* $1 \leqslant p < \infty$, $\gamma(p) \leqslant \mu < \nu + 2l + 3/2$ *where* $\mu = 3/2 - \nu - 2l$. *Then* $H_\nu(\mathscr{L}_{\mu,p})$ *is a proper subset of* $H_{\nu+2l}(\mathscr{L}_{\mu,p})$.

Proof. From [6], $H_\nu(\mathscr{L}_{\mu,p})$ is a subset of $H_{\nu+2l}(\mathscr{L}_{\mu,p})$. Note that since $0 < \nu + 2l < 1$, then $1/2 < \mu < 3/2$. Let $\eta = \nu + 2l$ and

$$f(x) = x^{-1/2}(x^2+1)^{-1/2}.$$

Then if $1 \leqslant p < \infty$,

$$\|f\|_{\mu,p} = \left\{\int_0^\infty (x^{\mu-1/2}(x^2+1)^{-1/2})^p \, dx/x\right\}^{1/p} < \infty,$$

since $1/2 < \mu < 3/2$. Hence $f \in \mathscr{L}_{\mu,p}$. From [2, 8.5(11)]

$$H_\eta(f)(x) = x^{1/2} I_{\eta/2}(x/2)\, K_{\eta/2}(x/2).$$

If $H_\nu(\mathscr{L}_{\mu,p}) = H_\eta(\mathscr{L}_{\mu,p})$, then from Theorem 6.3

$$\int_{\to 0}^{\to\infty} x^{\nu+2l-1} I_{\eta/2}(x/2)\, K_{\eta/2}(x/2)\, dx$$

converges and equals zero, which is impossible since the integrand is positive a.e.

THEOREM 6.5. *Suppose* $1 < p < \infty$, $\mu = 3/2 - \nu - 2l$, *and* $0 < \nu + 2l \leqslant 3/2 - \gamma(p)$. *Then a function g is in* $H_\nu(\mathscr{L}_{\mu,p})$ *if and only if*

(a) $g \in H_{\nu+2l}(\mathscr{L}_{\mu,p})$,

(b) $\int_{\to 0}^{1} t^{-\mu} g(t)\, dt$ *converges, and*

(c) $\phi \in H_{\nu+2l}(\mathscr{L}_{\mu,p})$ *where* $\phi(x) = x^{\mu-1}\int_{\to 0}^{x} t^{-\mu} g(t)\, dt$.

Proof. Suppose first that $g \in H_\nu(\mathscr{L}_{\mu,p})$, say $g = H_\nu f$, where $f \in \mathscr{L}_{\mu,p}$. Then since $\gamma(p) \leqslant \mu < \nu + 2l + 3/2$, (a) follows from [6], while (b) follows from Theorem 6.3. To prove (c), note that, with h as in the proof of Theorem 6.3, $g = Th$, and the inversion formula for T given in [4, (3.6)] yields

$$h(x) = -g(x) + (2\nu + 4l - 2)\,\phi(x) \tag{6.3}$$

for almost all positive x. Since g and $h \in H_{\nu+2l}(\mathscr{L}_{\mu,p})$, (c) follows.

Conversely, suppose (a), (b), and (c) are true and let h be defined by (6.3). Then from (a) and (c), $h \in H_{\nu+2l}(\mathscr{L}_{\mu,p})$, say $h = H_{\nu+2l}\psi$ with $\psi \in \mathscr{L}_{\mu,p}$. Since each T_r, $1 \leqslant r \leqslant l-1$, is a bijection of $\mathscr{L}_{\mu,p}$, we can find $f \in \mathscr{L}_{\mu,p}$ so that $\psi = T_{l-1}T_{l-2} \cdots T_1 f$. Hence $h = H_{\nu+2l}T_{l-1}T_{l-2} \cdots T_1 f$, and from [4, (3.6)], (6.3) and the definition of ϕ, $g = Th$, so that from Lemma 6.2, $g = H_\nu f$.

THEOREM 6.6. *If* $1 < p < \infty$, $0 < \nu + 2l < 3/2 - \gamma(p)$ *and* $f \in \mathscr{L}_{\mu,p}$, *where* $\mu = 3/2 - \nu - 2l$, *then for almost all* $x > 0$,

$$f(x) = x^{\nu+1/2}\left\{x^{-1}\frac{d}{dx}\right\}^2 x^{3/2-\nu}\int_0^{\to\infty} (xt)^{1/2} \times J_{\nu-2,l+2}(xt)(H_\nu f)(t)\,dt/t^2. \tag{6.4}$$

Proof. Choose ε so that $0 < \varepsilon < \nu + 2l < 3/2 - \gamma(p) - \varepsilon$. Then with $\mu_1 = 3/2 - \nu - 2l - \varepsilon$ and $\mu_2 = 3/2 - \nu - 2l + \varepsilon$, we have $\gamma(p) < \mu_i < 3/2 + \nu + 2l$, $i = 1, 2$.

Now let $f_1 = f\chi_{(1,\infty)}$, where χ_E denotes the characteristic function of the set E, and let $f_2 = f - f_1$. Then $f_i \in \mathscr{L}_{\mu_i,p}$ for $i = 1, 2$,, and if $g_i = H_\nu f_i$, (5.2) and (5.7) give respectively,

$$f_1(x) = x^{\nu+1/2}\left\{x^{-1}\frac{d}{dx}\right\}^2 x^{3/2-\nu}\int_0^{\infty} (xt)^{1/2} J_{\nu-2,l+2}(xt)\, g_1(t)\,dt/t^2 \tag{6.5}$$

and

$$f_2(x) = x^{\nu+1/2}\left\{x^{-1}\frac{d}{dx}\right\}^2 x^{3/2-\nu}\int_0^{\infty} (xt)^{1/2} J_{\nu-2,l+1}(xt)\, g_2(t)\,dt/t^2 \tag{6.6}$$

for almost all $x > 0$. But from Theorem 6.3,

$$\int_{\to 0}^{\to\infty} t^{\nu+2l-3/2} g_2(t)\,dt = 0,$$

and thus since the first term in $(xt)^{1/2} J_{\nu-2,l+1}(xt)/t^2$ is $Ax^{\nu+2l+1/2}t^{\nu+2l-3/2}$ for a certain constant A, it follows that

$$\int_0^{\infty} (xt)^{1/2} J_{\nu-2,l+1}(xt)\, g_2(t)\,dt/t^2$$

$$= \int_{\to 0}^{\to\infty} (xt)^{1/2} J_{\nu-2,l+2}(xt)\, g_2(t)\,dt/t^2,$$

and thus adding (6.5) and (6.6) and noting that the arrow isn't necessary at the lower limit, we obtain the result.

COROLLARY 6.7. *Under the same hypotheses,*

$$f(x) = x^{\nu+1/2}\left\{x^{-1}\frac{d}{dx}\right\}^2 x^{3/2-\nu}\int_{\to 0}^{\infty}(xt)^{1/2} \times J_{\nu-2,l+1}(xt)(H_\nu f)(t)\,dt/t^2. \tag{6.7}$$

Proof. Theorem 6.3 shows that the integrals on the right of (6.4) and (6.7) are equal, and it is easy to see that the integral in (6.7) does not need the arrow in the upper limit.

LEMMA 6.8. *Suppose that* $1 < p < \infty$, $\gamma(p) = \mu < \nu + 2l + 3/2$, *where* $\mu = 3/2 - \nu - 2l$, *and* $f \in \mathscr{L}_{\mu,p}$. *Then*

$$\int_1^{\infty} t^{1/2-\nu-2l} f(t)\,dt \int_{at}^{\to\infty} u^{\nu+2l-1} J_{\nu,l}(u)\,du \to 0$$

as $a \to \infty$.

Proof. Since $\mu = 3/2 - \nu - 2l$ and $\mu = \gamma(p) < 1$, it follows that $\nu + 2l - 1 = 1/2 - \mu > -1/2$ and hence for large positive u,

$$J_{\nu+1,l}(u) = O(u^{-1/2}) + O(u^{\nu+2l-1}) = O(u^{\nu+2l-1}). \tag{6.8}$$

From [5, (2.1)]

$$\frac{d}{du}(u^{\nu+1}J_{\nu+1,l}(u)) = u^{\nu+1}J_{\nu,l}(u),$$

and thus, on integrating by parts,

$$\int_{at}^{\to\infty} u^{2l-2}\cdot u^{\nu+1}J_{\nu,l}(u)\,du$$

$$= -(at)^{\nu+2l-1}J_{\nu+1,l}(at) - (2l-2)\int_{at}^{\infty} u^{\nu+2l-2}J_{\nu+1,l}(u)\,du, \tag{6.9}$$

the existence of the integral on the right at infinity, and the vanishing of the integrated term, following from (6.8) since $\nu + 2l < 1$.

By Holder's inequality

$$\left|\int_1^\infty t^{1/2-\nu-2l} f(t)(at)^{\nu+2l-1} J_{\nu+1,l}(at)\,dt\right|$$

$$\leqslant a^{\nu+2l-1}\left\{\int_1^\infty |t^{3/2-\nu-2l} f(t)|^p\,dt/t\right\}^{1/p}$$

$$\cdot\left\{\int_1^\infty |t^{\nu+2l-1} J_{\nu+1,l}(at)|^{p'}\,dt/t\right\}^{1/p'}$$

$$\leqslant \|f\|_{\mu,p}\left\{\int_a^\infty |u^{\nu+2l-1} J_{\nu+1,l}(u)|^{p'}\,du/u\right\}^{1/p'}, \tag{6.10}$$

and this tends to zero as $a \to \infty$.

Also, from (6.8), if a is large and positive,

$$\int_1^\infty t^{1/2-\nu-2l}|f(t)|\,dt \int_{at}^\infty u^{\nu+2l-2}|J_{\nu+1,l}(u)|\,du$$

$$\leqslant \int_1^\infty t^{1/2-\nu-2l}|f(t)|\,dt \int_{at}^\infty K u^{2(\nu+2l)-3}\,du$$

$$= La^{2(\nu+2l-1)}\int_1^\infty t^{\nu+2l-3/2}\,|f(t)|\,dt$$

$$= La^{2(\nu+2l-1)}\int_1^\infty |t^\mu f(t)|\,t^{1-2\mu}\,dt/t$$

$$\leqslant La^{2(\nu+2l-1)}\|f\|_{\mu,p}\left\{\int_1^\infty t^{(1-2\mu)p'-1}\,dt\right\}^{1/p'}, \tag{6.11}$$

which tends to zero as $a \to \infty$ since $\nu + 2l < 1$.

The lemma now follows from (6.9), (6.10), and (6.11).

THEOREM 6.9. *Suppose* $1 < p < \infty$, $1/2 < \nu + 2l \leqslant 3/2 - \gamma(p)$, *and* $f \in \mathscr{L}_{\mu,p}$, *where* $\mu = 3/2 - \nu - 2l$. *Then for almost all* $x > 0$,

$$f(x) = -x^{\nu-1/2}\frac{d}{dx}x^{1/2-\nu}\int_0^{\to\infty}(xt)^{1/2}$$

$$\times J_{\nu-1,l+1}(xt)(H_\nu f)(t)\,dt/t. \tag{6.12}$$

Proof. If $1/2 < \nu + 2l < 3/2 - \gamma(p)$, the proof follows that of Theorem 6.6 exactly, but appealing to (5.3) and (5.8) rather than (5.2) and (5.7).

In the case when $\nu+2l=3/2-\gamma(p)$, we can still obtain (6.12) for f_2 from (5.8) for f_2, and our proof will be complete if we establish

$$f_1(x)=-x^{\nu-1/2}\frac{d}{dx}x^{1/2-\nu}\int_0^{\to\infty}(xt)^{1/2}J_{\nu-1,l+1}(xt)\,g_1(t)\,dt/t \quad (6.13)$$

for almost all $x>0$.

Let $x>0$ and $a>1$, and let

$$\begin{aligned}\psi_x(t)&=t^{-1/2}J_{\nu-1,l+1}(xt), && 0<t\leqslant a,\\ &=t^{-1/2}J_{\nu-1,l}(xt), && t>a.\end{aligned}$$

Then if we choose the constant A suitably,

$$\psi_x(t)=t^{-1/2}(J_{\nu-1,l+1}(xt)-A(xt)^{\nu+2l-1}\,\chi_{(a,\infty)}(t)) \quad (6.14)$$

for $t>0$.

For large positive t, as we have shown earlier,

$$J_{\nu-1,l}(xt)=O(t^{-1/2})+O(t^{\nu+2l-3}),$$

and the first term dominates since $\nu+2l<1$, and thus $\psi_x(t)=O(t^{-1})$ as $t\to\infty$. Also, as $t\to 0$, $\psi_x(t)=O(t^{\nu+2l+1/2})$, and thus it follows that $\psi_x\in\mathscr{L}_{3/2-\nu-2l,p'}=\mathscr{L}_{\gamma,p'}$, where $\gamma=\gamma(p)$. Hence, by Lemma 3.3,

$$\int_0^\infty f_1(t)(H_\nu\psi_x)(t)\,dt=\int_0^\infty\psi_x(t)(H_\nu f_1)(t)\,dt, \quad (6.15)$$

and we can split the right-hand side of (6.15) into

$$\int_0^{\to\infty}t^{-1/2}J_{\nu-1,l+1}(xt)\,g_1(t)\,dt-Ax^{\nu+2l-1}\int_a^{\to\infty}t^{\nu+2l-3/2}g_1(t)\,dt,$$

since the last integral was shown to converge in Theorem 6.3.

To use (6.15) we need to compute $(H_\nu\psi_x)(t)$. But we showed in Lemma 3.2 that $(H_\nu g_x)(t)=-x^{1-\nu}t^{-1/2}J_{\nu-1,l+1}(xt)$, and thus this function is in $\mathscr{L}_{1/2,2}=L_2(0,\infty)$. Hence since from [5, Sect. 3], H_ν is self inverting on $L_2(0,\infty)$, the Hankel transformation of the first term in (6.14) is $-x^{\nu-1}g_x$. Note that since $\nu+2l<1$, the second term in (6.14) is in $L_2(0,\infty)$, and thus from [5, (1a)], for almost all $t>0$, its Hankel transform is given by the mean square limit as $R\to\infty$ of

$$x^{\nu+2l-1}t^{1/2}\int_a^R u^{\nu+2l-1}J_{\nu,l}(tu)\,du.$$

But

$$\int_{at}^{\to\infty} u^{\nu+2l-1} J_{\nu,l}(u)\, du = t^{\nu+2l} \int_{a}^{+\infty} u^{\nu+2l-1} J_{\nu,l}(tu)\, du,$$

the convergence of the integral for $t > 0$, $a > 1$ having been shown in the proof of Lemma 6.8. Hence using the formula for g_x we must have for almost all $t > 0$,

$$(H_\nu \psi_x)(t) = -x^{\nu-1} t^{1/2-\nu} \chi_{(0,x)}(t) - A t^{1/2-\nu-2l} \times x^{\nu+2l-1} \int_{at}^{\to\infty} u^{\nu+2l-1} J_{\nu,l}(u)\, du.$$

Inserting this expression into (6.15), we find that

$$-x^{\nu-1} \int_0^x t^{1/2-\nu} f_1(t)\, dt - A x^{\nu+2l-1} \times \int_1^\infty t^{1/2-\nu-2l} f(t)\, dt \int_{at}^{\to\infty} u^{\nu+2l-1} J_{\nu,l}(u)\, du$$
$$= \int_0^{\to\infty} t^{-1/2} J_{\nu-1,l+1}(xt)\, g_1(t)\, dt - A x^{\nu+2l-1} \int_a^{\to\infty} t^{\nu+2l-3/2} g_1(t)\, dt.$$

If we let $a \to \infty$, from Lemma 6.8 the second term on the left-hand side tends to zero, as does the second term on the right-hand side, and thus

$$\int_0^x t^{1/2-\nu} f_1(t)\, dt = -x^{1-\nu} \int_0^{\to\infty} t^{-1/2} J_{\nu-1,l+1}(xt)\, g_1(t)\, dt,$$

from which (6.13) follows on differentiation.

COROLLARY 6.10. *Under the conditions of Theorem* 6.9,

$$f(x) = -x^{\nu-1/2} \frac{d}{dx} x^{1/2-\nu} \int_{\to 0}^{\infty} (xt)^{1/2} J_{\nu-1,l}(xt)(H_\nu f)(t)\, dt/t.$$

Proof. Much the same as the proof of Corollary 6.7.

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